

# Slow motion for compressible isentropic Navier–Stokes equations

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**Abstract.** We consider the compressible Navier–Stokes equations for isentropic dynamics with real viscosity on a bounded interval. In the case of boundary data defining an admissible shock wave for the corresponding inviscid hyperbolic system, we determine a scalar differential equation describing the motion of the internal transition layer. In particular, for viscosity  $\varepsilon$  small, the velocity of the motion is exponentially small. The approach is based on the construction of a 1-parameter manifold of approximate solutions and on an appropriate projection of the evolution of the complete Navier–Stokes system towards such manifold.

**Keywords.** Navier–Stokes equations; Metastability; Hyperbolic–parabolic systems.

**AMS subject classifications.** Primary 76N99; Secondary 35B25, 35Q35.

## 1. INTRODUCTION

This article is devoted to the description of slow-motion for the hyperbolic-parabolic Navier–Stokes system for compressible isentropic fluid with real viscosity, that is, in term of the variables density/velocity  $(\rho, w)$

$$(1) \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho w)}{\partial x} = 0, \quad \frac{\partial(\rho w)}{\partial t} + \frac{\partial}{\partial x} \left\{ \rho w^2 + P(\rho) - \varepsilon \nu(\rho) \frac{\partial w}{\partial x} \right\} = 0.$$

The functions defining the pressure  $P$  and the viscosity  $\nu$  are required to satisfy the standard assumptions

$$P'(\rho) > 0, \quad P''(\rho) > 0, \quad \nu(\rho) > 0$$

for any  $\rho$  under consideration. The relevant cases  $P(\rho) = C\rho^\alpha$  with  $\alpha > 1$  and  $\nu(\rho) = C\rho^\beta$  with  $\alpha, \beta \geq 1$  and  $C > 0$  fit into the general framework. In particular, shallow water Saint–Venant system with viscosity corresponds to the case  $\alpha = 2, \beta = 1$  (see [8] for the derivation of the model and [5] for a recent review on shallow water equations).

Given  $\ell > 0$ , the space variable  $x$  belongs to the bounded interval  $(-\ell, \ell)$  and system (1) is complemented with boundary conditions

$$\rho(-\ell) = \rho_- \quad w(\pm\ell) = w_\pm > 0.$$

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Let us set

$$F(u, v) = \frac{v^2}{u} + P(u).$$

Considering the variables density/momentum  $(u, v) = (\rho, \rho w)$ , system (1) becomes

$$(2) \quad \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left\{ F(u, v) - \varepsilon v(u) \frac{\partial}{\partial x} \left( \frac{v}{u} \right) \right\} = 0,$$

with the boundary conditions

$$(3) \quad w_{\pm} u(\pm \ell) - v(\pm \ell) = 0, \quad v(-\ell) = \rho_- w_-$$

In the limiting case  $\ell \rightarrow +\infty$ , system (2) is known to support traveling wave solutions, i.e. solution with the form  $(u, v) = (U(x - ct), V(x - ct))$  satisfying the asymptotic conditions

$$(U, V)(-\infty) = (u_-, v_-), \quad (U, V)(+\infty) = (u_+, v_+)$$

for appropriate choices of  $(u_{\pm}, v_{\pm})$  and  $c$  given by the usual Rankine–Hugoniot relation. Thanks to the galileian invariance of (2), we may assume, without loss of generality, the speed  $c$  to be zero. In such a case, the component  $V$  turns to be constant, so that  $v_+$  and  $v_-$  are forced to be equal to a common value, denoted here by  $v_*$ . Hence, the equivalent form of (3) for the special solution  $(U, V)$  is

$$U(\pm \ell) = u_{\pm} := \frac{v_*}{w_{\pm}}, \quad V(-\ell) = v_*$$

The wave solution  $(U, V)$  belongs to a 1-parameter family of stationary solutions to (2) generated by the space translation group. The stability analysis of such family has been explored for a long time and a number of orbital stability results for different regimes and structure functions has been proved (see [20, 1, 10, 21]).

The analysis of the dynamics in bounded domains has been also investigated (among others, we quote [4, 16, 17]), but always with a limited attention to the dynamical behavior close to equilibrium configurations. In term of special solution, in bounded intervals  $(-\ell, \ell)$  with fixed  $\ell > 0$ , the situation is different with respect to case of the whole real line. Formally setting  $\varepsilon = 0$  in (2), there still exists a one-parameter family of steady states given by a sharp transition at some point  $\xi \in (-\ell, \ell)$ . Differently, for  $\varepsilon > 0$ , there exists a single steady state satisfying the boundary conditions (3) with  $w_{\pm}, v_-$  (or, equivalently,  $u_{\pm}, v_*$ ) chosen so that there exists a stationary wave in the whole real line with same asymptotic states.

As a consequence, in analogy to the case of scalar viscous conservation laws, it is expected that, in the regime of  $\varepsilon$  small, the solution determined by an initial datum consisting of a single transition from  $(u_-, v_*)$  to  $(u_+, v_*)$  converges in a short timescale to a specific profile with transition located at some point  $\xi = \xi(t)$  and, then, on a much longer timescale, moves to the location of the single steady state. Such motion is an effect of the boundary data and it is expected to be very slow. Our aim is to derive formally a differential equation for the location  $\xi$  and to show that the motion of  $\xi$  is indeed exponentially slow for  $\varepsilon \approx 0$ . The first tool is the definition of an *approximate invariant manifold*  $\{W(\cdot, \xi) : \xi \in (-\ell, \ell)\}$  whose elements are approximate steady state of (2) and resemble, in a sense,

transitions from  $(u_-, v_*)$  to  $(u_+, v_*)$  located at  $\xi$ . Among the many different and significant choices for constructing such manifold, our preference goes to the one we explored in [19] for the case of scalar conservation laws and that consists in matching exact steady states in  $(-\ell, \xi)$  and in  $(\xi, \ell)$  at  $x = \xi$  by imposing some appropriate conditions. Assuming that the spectrum of the linear operator  $\mathcal{L}_\xi$ , obtained by linearizing (2) at  $W(\cdot, \xi)$ , has a first eigenvalue that is real and simple, applying a projection method in the spirit of [23], we determine the equation (see Section 3)

$$\frac{d\xi}{dt} = -\frac{\psi(\xi, \xi) \kappa_+(\xi) - \kappa_-(\xi)}{\phi(\xi, \xi) u_+ - u_-}.$$

where  $(\phi(\cdot, \xi), \psi(\cdot, \xi))$  is the first eigenfunction of the adjoint operator  $\mathcal{L}_\xi^*$  and the functions  $\kappa_\pm$  are implicitly defined by

$$\int_{u_*}^{u_\pm} \frac{v_* v(s)}{s^2(\kappa_\pm - F(s, v_*))} ds = \pm \frac{\ell \mp \xi}{\varepsilon}$$

where  $u_*$  is the value determined by the condition  $P'(u_*)u_*^2 = v_*^2$ . The difference function  $\xi \mapsto \kappa_+(\xi) - \kappa_-(\xi)$  is continuous, monotone increasing, diverges at  $\pm\infty$  as  $\xi \rightarrow \pm\ell$ ; hence, the above differential equation possesses a single equilibrium point  $\xi_*$ , corresponding to the unique stationary solution of the problem.

For small  $\varepsilon$ , the elements of the approximate manifolds tend to a piecewise constant configuration, with a single jump located at  $\xi$ . Thus, it is possible to determine the leading term in the expression for the eigenfunction  $(\phi, \psi)$  of the adjoint operator  $\mathcal{L}_\xi^*$  and to obtain a new version of the motion equation

$$\frac{d\xi}{dt} = -\frac{(\xi + \ell)\lambda_1(\xi) \kappa_+(\xi) - \kappa_-(\xi)}{\partial_u F(u_-, v_*) u_+ - u_-}.$$

where  $\lambda_1(\xi)$  is the first eigenvalue of the operator  $\mathcal{L}_\xi$ . Let us stress that, at such step, the choice of the boundary conditions is particularly relevant, since it determines the specific structure of the eigenfunction  $(\phi, \psi)$ .

As it should be, the behavior of  $\lambda_1$  for small  $\varepsilon$  plays a crucial rôle (details are given in Section 3). Extrapolating from [14], that concerns with general systems of conservation laws with a (non-physical) second-order parabolic term, and by the quoted results on asymptotic stability of steady waves on the whole real line, we expect that  $\lambda_1$  is (exponentially) small and negative. We are not aware of any available rigorous result in this sense at the present time.

Assuming that such eigenvalue stays bounded in the limit  $\varepsilon \rightarrow 0^+$ , the speed of motion along the approximate manifold, is determined by the functions  $\kappa_\pm$ , whose leading terms can be determined starting from their definitions, obtaining the final form for the motion's equation

$$\frac{d\xi}{dt} = -\left\{ \frac{\partial_u F^+}{\partial_u F^-} \frac{u_+ - u_*}{u_+ - u_-} \exp\left(-\frac{\partial_u F^+}{\partial_u G^+} \frac{\ell - \xi}{\varepsilon}\right) + \frac{u_* - u_-}{u_+ - u_-} \exp\left(\frac{\partial_u F^-}{\partial_u G^-} \frac{\xi + \ell}{\varepsilon}\right) \right\} (\xi + \ell) \lambda_1(\xi)$$

where  $\partial_u G(u, v) = v v(u)/u^2$  and the upper scripts  $\pm$  indicate that the function is calculated at  $(u_\pm, v_*)$ . In particular, for small  $\varepsilon$ , the motion is exponentially slow. The stability of the equilibrium point  $\xi_*$  is still encoded in the sign of the first eigenvalue  $\lambda_1$  that the present analysis is not able to reveal.

This paper follows the research line on metastable behaviors for conservation laws widely explored in the last decades. The first contribution has been the pioneering article [13] concerning the analysis of the scalar Burgers equation, that has been also the subject of [23] (based on the use of *projection method* and *WKB expansions*) and [15] (standing on an adapted version of the *method of matched asymptotics expansion*). A rigorous analysis has been performed in [6, 7]), where the one-parameter family of reference functions is chosen as a family of traveling wave solutions to the viscous equation satisfying the boundary conditions and with non-zero velocity. Slow motion for the viscous Burgers equation in unbounded domains has been also considered in literature: the case of the half-line  $(0, +\infty)$  has been treated in [24, 18, 22]; while the case of whole real line has been examined in [12, 11, 3] (with emphasis on the generation of  $N$ -wave like structures and their evolution towards nonlinear diffusion waves).

Despite of the wide number of contributions to the stability of traveling profiles in the whole real line, results relative to slow motion and metastable behavior in the case of systems of conservation laws appear to be rare. We are only aware of [9] (that uses asymptotic expansions to deal with systems of conservation laws, with model examples being the Navier-Stokes equations of compressible viscous heat conductive fluid and the Keyfitz-Kranzer system, arising in elasticity), [14] (that deals with the problem of proving convergence to a stationary solution for a system of conservation laws with viscosity, with an approach based on the analysis of the linearized operator at the steady state) and [2] (that addresses to the Saint-Venant equations for shallow water and, precisely, the phenomenon of formation of roll-waves, by means of a combination of analytical techniques and numerical results).

In this respect, we consider our contribution, even if mainly based on formal arguments, original and hopefully stimulating for people working in the area of dynamical properties of solutions to systems of conservation laws. Specifically, it seems that the equation we propose for the motion of the transition layer  $\xi$  (that is the slow dynamics along the approximate equilibrium manifold) is the first attempt in this direction for isentropic Navier–Stokes equations for compressible fluids.

The article is divided into three more Sections. In Section 2, we present the general procedure to derive formal equation for the motion along an approximate equilibrium manifold in the case of general hyperbolic–parabolic systems. For pedagogical reasons, we also show how the approach simplifies in the case of scalar Burgers equation. Section 3 is the heart of the paper. It deals with isentropic Navier–Stokes equation for compressible fluids, written in the form (2). After recasting the admissibility condition for entropic jumps in the unviscous case, we build up the approximate equilibrium manifold, working with time-independent solutions and matching them at  $x = \xi$  by means of appropriate transmission condition. Using such special approximate solutions, we are able to determine an equation describing the evolution along the manifold. In order to determine an explicit expression for the ratio  $\psi(\xi, \xi)/\phi(\xi, \xi)$  appearing in such equation for the motion along the manifold, we analyze the eigenvalue problem for the adjoint operator and we deduce an approximated version of its solutions for the regime  $\varepsilon$  small, by approximating the element of the manifold to be piecewise

constant functions with a single jump located at  $\xi$ . Finally, in order to get an ultimate version of the motion's equation, we determine the leading term for the functions  $\kappa_{\pm}$  in the limit  $\varepsilon \rightarrow 0^+$ , Section 4 contains the conclusions and it is mainly dedicated to propose a number of eventual research direction motivated by the present work.

## 2. SCALAR REDUCED DYNAMICS FOR HYPERBOLIC-PARABOLIC SYSTEMS

As a first step, we present the strategy to obtain approximate equation for the motion along an approximate manifold of solutions to a general hyperbolic-parabolic system having the form

$$(4) \quad \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} \left\{ f(w) - \varepsilon b(w) \frac{\partial w}{\partial x} \right\} = 0,$$

where  $w = w(x, t) \in \mathbb{R}^n$ ,  $x \in I$ ,  $t > 0$ . Our target is to apply such approach to the case of isentropic Navier-Stokes equation. Thus, for the sake of simplicity, we do not state precise assumptions on the structure functions  $f$  and  $b$  and we proceed in a purely formal way. In any case, we expect the procedure to be meaningful for the usual class of hyperbolic-parabolic systems considered in the recent literature (see [25] and descendants).

Given  $T > 0$ , we consider the initial-boundary value problem for (4) determined by the conditions

$$(5) \quad w|_{t=0} = w_0$$

complemented with appropriate boundary conditions.

Given an open interval  $J$  and a one-parameter family of functions

$$\{W(\cdot; \xi) \in [H^1(I)]^n : \xi \in J\}$$

satisfying the boundary conditions, let  $\xi \mapsto R^\varepsilon(\cdot; \xi)$  be the distribution-valued map defined by

$$(6) \quad \langle R(\cdot; \xi), \varphi \rangle := - \int_I \left\{ \varepsilon b(W) \frac{\partial W}{\partial x} - f(W) \right\} \cdot \frac{d\Phi}{dx} dx$$

for any continuously differentiable function  $\Phi : I \rightarrow \mathbb{R}^n$  with  $\Phi(\pm\ell) = 0$ . In what follows, we call  $R(\cdot; \xi)$  the residual of  $W(\cdot; \xi)$  with respect to equation (4). The family  $\{W(\cdot; \xi)\}$  is considered as an *approximate invariant manifold* for (4), in the sense that the residuals  $R(\cdot; \xi)$  vanishes as  $\varepsilon \rightarrow 0^+$  (in a sense to be made precise) for any  $\xi \in J$ .

Next, we look for solutions to (4)–(5) in the form

$$w(\cdot, t) = W(\cdot; \xi(t)) + z(\cdot, t)$$

with unknown  $\xi = \xi(t)$  and  $z = z(\cdot, t)$  to be determined. Substituting into (4) and disregarding the nonlinear terms in  $v$ , we obtain an approximated equation for the perturbation  $z$

$$(7) \quad \frac{\partial z}{\partial t} = \mathcal{L}_\xi z + R(\cdot; \xi) - \frac{\partial W}{\partial \xi}(\cdot; \xi) \frac{d\xi}{dt}$$

where  $R(\cdot; \xi)$  is the distribution defined in (6) and  $\mathcal{L}_\xi$  is the linearized operator at  $W(\cdot; \xi)$ , i.e.

$$\mathcal{L}_\xi z := \frac{\partial}{\partial x} \left\{ \varepsilon b(W) \frac{\partial z}{\partial x} + db(W) z \frac{\partial W}{\partial x} - df(W) z \right\}$$

where we use the notation

$$(db(W)zw)_i := \sum_{j,k} \frac{\partial b_{ij}}{\partial w_k}(W) z_k w_j \quad \text{with } b = (b_{ij}).$$

Next, let us assume that, for any  $\xi \in I$ , the operator  $\mathcal{L}_\xi$  has a first eigenvalue  $\lambda_1(\xi)$  that is real and simple. Let  $\mathcal{L}_\xi^*$  be the adjoint operator of  $\mathcal{L}_\xi$

$$\mathcal{L}_\xi^* z := \frac{\partial}{\partial x} \left\{ \varepsilon b(W)^t \frac{\partial z}{\partial x} \right\} - \left( Db(W) \frac{\partial W}{\partial x} - df(W)^t \right) \frac{\partial z}{\partial x}$$

where  $^t$  denote the transpose and

$$(Db(W)wz)_k := \sum_{i,j} \frac{\partial b_{ij}}{\partial w_k}(W) z_i w_j \quad \text{with } b = (b_{ij}).$$

Given  $\ell > 0$ , set  $I := (-\ell, \ell)$  and

$$\langle u, v \rangle := \int_{-\ell}^{\ell} u(x) \cdot v(x) dx \quad u, v \in [L^2(I)]^n,$$

where  $\cdot$  denotes the usual scalar product in  $\mathbb{R}^n$ . Denoting by  $\omega_1 = \omega_1(\cdot; \xi)$  an eigenfunction of the adjoint operator relative to the first eigenvalue and setting

$$z_1 = z_1(\xi; t) := \langle \omega_1(\cdot; \xi), z(\cdot, t) \rangle,$$

we determine a differential equation for the function  $t \mapsto \xi(t)$  by imposing that the component  $z_1$  is identically zero, that is

$$\frac{d}{dt} \langle \omega_1(\cdot; \xi(t)), z(\cdot, t) \rangle = 0 \quad \text{and} \quad \langle \omega_1(\cdot; \xi_0), z(\cdot, 0) \rangle = 0.$$

Using equation (7), we infer

$$\langle \omega_1(\cdot; \xi), \mathcal{L}_\xi z + R(\cdot; \xi) - \frac{\partial W}{\partial \xi}(\cdot; \xi) \frac{d\xi}{dt} \rangle + \left\langle \frac{\partial \omega_1}{\partial \xi} \frac{d\xi}{dt}, z \right\rangle = 0$$

Since  $\langle \omega_1, \mathcal{L}_\xi z \rangle = \lambda_1 \langle \omega_1, z \rangle$ , we obtain a scalar differential equation for the variable  $\xi$ , the latter equation can be rewritten as

$$\left\{ \left\langle \omega_1, \frac{\partial W}{\partial \xi} \right\rangle - \left\langle \frac{\partial \omega_1}{\partial \xi}, z \right\rangle \right\} \frac{d\xi}{dt} = \langle \omega_1, R \rangle$$

to be considered together with the condition on the initial datum  $\xi_0$

$$\langle \omega_1(\cdot; \xi_0), z(\cdot, 0) \rangle = 0.$$

Neglecting the second term in the coefficient of the derivative of  $\xi$ , we end up with an (approximated) equation for the motion along the manifold  $\{W(\cdot; \xi)\}$

$$(8) \quad \langle \omega_1(\cdot; \xi), \frac{\partial W}{\partial \xi}(\cdot; \xi) \rangle \frac{d\xi}{dt} = \langle \omega_1(\cdot; \xi), R(\cdot; \xi) \rangle.$$

Our next effort is to determine a practical version of the above equation in the limiting regime  $\varepsilon \rightarrow 0^+$  and for the specific case of hyperbolic–parabolic systems (4).

First of all, given  $\xi \in I = (-\ell, \ell)$ , we require the function  $W(\cdot; \xi)$  to converge in the limit  $\varepsilon \rightarrow 0^+$  to the step function jumping at  $x = \xi$  from  $w_-$  to  $w_+$

$$(9) \quad \lim_{\varepsilon \rightarrow 0^+} W(x, \xi) = w_- \chi_{(-\ell, \xi)}(x) + w_+ \chi_{(\xi, \ell)}(x)$$

If the limit is in the sense of distributions, approximating the increment ratio for  $h > 0$  by

$$\frac{W(x; \xi + h) - W(x, \xi)}{h} \approx -\frac{1}{h} [w] \chi_{(\xi, \xi+h)}(x)$$

we infer the asymptotic representation

$$\frac{\partial W}{\partial \xi}(\cdot; \xi) = -[w] \delta_\xi(\cdot) + o(1) \quad \text{as } \varepsilon \rightarrow 0^+,$$

where  $[w] := w_+ - w_-$  and  $\delta_\xi$  is the Dirac distribution concentrated at  $\xi$ . Thus, for small  $\varepsilon$ , there hold

$$\langle \omega_1(\cdot; \xi), \frac{\partial W}{\partial \xi}(\cdot; \xi) \rangle = -\omega_1(\xi; \xi) \cdot [w].$$

Finally, assuming  $W$  to be chosen so that  $R$  is a Dirac distribution concentrated at  $\xi$ ,

$$(10) \quad R(x, \xi) = r(\xi) \delta_\xi(x)$$

for some function  $r = r(\xi)$ , we deduce our final expression for the reduced dynamics (8) along the manifold  $\{W(\cdot; \xi)\}$

$$(11) \quad \frac{d\xi}{dt} = \theta(\xi) := -\frac{\omega_1(\xi; \xi) \cdot r(\xi)}{\omega_1(\xi; \xi) \cdot [w]}.$$

In order to make equation (11), it is necessary to determine the specific expression of the function  $r$ , describing the residual of  $W(\cdot; \xi)$ , and the ratio between the components of the first eigenfunction  $\omega_1$  of the adjoint operator in the direction of  $r$  and in the direction of the jump  $[w]$ .

**Viscous Burgers equation.** Let us consider the case of the scalar Burgers equation with viscosity

$$(12) \quad \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} \left\{ \frac{1}{2} w^2 - \varepsilon \frac{\partial w}{\partial x} \right\} = 0.$$

with boundary conditions

$$w(-\ell, t) = \bar{w}, \quad w(+\ell, t) = -\bar{w}$$

for some given  $\bar{w} > 0$ . In this case  $[w] = -2\bar{w}$  and equation (11) reduces to

$$(13) \quad \frac{d\xi}{dt} = \frac{r(\xi)}{2\bar{w}},$$

where  $r(\xi)$  is the residual of  $W(\cdot; \xi)$ .

In order to satisfy the requirement (10), we consider a specific approximate invariant manifold: for  $\xi \in I$ , we build  $W(x, \xi)$  by matching two steady states of the equation with appropriate boundary conditions. Namely, we set

$$W(x, \xi) := \begin{cases} W_-(x, \xi) & -\ell < x < \xi < \ell \\ W_+(x, \xi) & -\ell < \xi < x < \ell, \end{cases}$$

where  $W_{\pm}$  are steady state of (12) in  $(-\ell, \xi)$  and  $(\xi, \ell)$ , such that

$$W_-(-\ell; \xi) = \bar{w}, \quad W_-(\xi; \xi) = W_+(\xi; \xi) = 0, \quad W_+(\ell; \xi) = -\bar{w}.$$

Functions  $W_{\pm}$  can be expressed by means of an implicit formula. For  $\bar{w} > 0$ , let us set

$$\Sigma := \{(w, \kappa) : w \in (-\bar{w}, \bar{w}), \kappa > w^2/2\}.$$

Then, defining the function  $\Gamma = \Gamma(w, \kappa)$  with  $(w, \kappa) \in \Sigma$  by

$$\Gamma(w; \kappa) := \int_0^w \frac{ds}{\kappa - s^2/2} = \sqrt{\frac{2}{\kappa}} \tanh^{-1} \left( \frac{w}{\sqrt{2\kappa}} \right),$$

functions  $W_{\pm}$  are implicitly given by

$$(14) \quad \varepsilon \Gamma(W_{\pm}(x, \xi), \kappa_{\pm}) = \xi - x.$$

where the values  $\kappa_{\pm} = \kappa_{\pm}(\xi)$  are uniquely determined by the conditions

$$(15) \quad \varepsilon \Gamma(\pm \bar{w}, \kappa_{\mp}) = \xi \pm \ell.$$

Since  $W_-$  and  $W_+$  are steady states of (12) in  $(-\ell, \xi)$  and  $(\xi, \ell)$ , respectively, we deduce, integrating by parts, that the residual  $R$  is

$$\begin{aligned} \langle R(\cdot; \xi), \Phi \rangle &= - \int_{-\ell}^{\xi} \left\{ \varepsilon \frac{\partial W_-}{\partial x} - \frac{1}{2} W_-^2 \right\} \frac{d\Phi}{dx} dx - \int_{\xi}^{\ell} \left\{ \varepsilon \frac{\partial W_+}{\partial x} - \frac{1}{2} W_+^2 \right\} \frac{d\Phi}{dx} dx \\ &= -\varepsilon \frac{\partial W_-}{\partial x}(\xi) \Phi(\xi) + \varepsilon \frac{\partial W_+}{\partial x}(\xi) \Phi(\xi) = \varepsilon \left[ \frac{\partial W}{\partial x} \right]_{\xi} \Phi(\xi) \end{aligned}$$

Differentiating (14) with respect to  $x$ , we get

$$(16) \quad \varepsilon \frac{\partial W_{\pm}}{\partial x}(x, \xi) = \frac{1}{2} W_{\pm}^2(x, \xi) - \kappa_{\pm},$$

thus, in the notation of (10), there holds

$$r(\xi) = \varepsilon \left[ \frac{\partial W}{\partial x} \right]_{\xi} = \varepsilon \frac{\partial W_+}{\partial x}(\xi; \xi) - \varepsilon \frac{\partial W_-}{\partial x}(\xi; \xi) = \kappa_-(\xi) - \kappa_+(\xi),$$

giving an “almost explicit” expression for (13).

Since we are considering the regime  $\varepsilon \rightarrow 0^+$ , it is possible to approximate the formulas defining  $\kappa_{\pm}$  and obtain a simpler o.d.e. describing the slow dynamics. Handling the explicit expression for the function  $\Gamma$ , conditions (15) become

$$\sqrt{2\kappa_{\mp}} \tanh \left\{ \frac{\ell \pm \xi}{\varepsilon} \sqrt{\frac{\kappa_{\mp}}{2}} \right\} = \bar{w}.$$

In particular, in the limit  $\varepsilon \rightarrow 0^+$ , we get

$$\lim_{\varepsilon \rightarrow 0^+} \kappa_{\pm}(\xi) = \frac{1}{2} \bar{w}^2.$$

Thus, as  $\varepsilon \rightarrow 0^+$ , there approximately holds

$$\kappa_{\mp} \approx \frac{\bar{w}^2}{2 \tanh \{ \bar{w}(\ell \pm \xi)/2\varepsilon \}^2} \approx \frac{1}{2} \bar{w}^2 \left\{ 1 + 4 \exp \left( -\frac{\bar{w}}{\varepsilon} (\ell \pm \xi) \right) \right\}$$



Collecting, we end up with the equation

$$\frac{d\xi}{dt} \approx \bar{w} \left\{ \exp\left(-\frac{\bar{w}}{\varepsilon}(\ell + \xi)\right) - \exp\left(-\frac{\bar{w}}{\varepsilon}(\ell - \xi)\right) \right\}$$

corresponding to the formula determined in [23].

### 3. COMPRESSIBLE ISENTROPIC NAVIER–STOKES EQUATIONS

Given smooth functions  $P = P(u)$  and  $\nu = \nu(u)$ , let us consider the hyperbolic-parabolic system for compressible isentropic fluids

$$(17) \quad \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \frac{\partial}{\partial x} \left( \frac{v}{u} \right) \right\} = 0.$$

where the pressure  $P$  is such that  $P'(u) > 0$  and  $P''(u) > 0$  and the viscosity  $\nu$  is such that  $\nu(u) > 0$  for any  $u$  under consideration.

System (17) is considered for  $x \in (-\ell, \ell)$  together with the boundary conditions

$$(18) \quad v_{\pm} u(\pm\ell) - u_{\pm} v(\pm\ell) = 0, \quad v(-\ell) = v_-$$

for some  $u_{\pm}, v_{\pm}$  with the value  $v_-$  being strictly positive.

**Admissible jumps in the vanishing viscosity limit.** As a first step, let us consider the limiting unviscous regime  $\varepsilon \rightarrow 0^+$ . Putting formally  $\varepsilon = 0$  in (17), we obtain the hyperbolic system for unviscous isentropic fluids

$$(19) \quad \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left\{ \frac{v^2}{u} + P(u) \right\} = 0.$$

Physical solutions are weak solutions to (19) satisfying appropriate jump conditions determined by the couple entropy/entropy flux

$$\mathcal{E}(u, v) := \frac{v^2}{2u} + \Pi(u), \quad \mathcal{Q}(u, v) := \frac{v^3}{2u^2} + \Pi'(u)v.$$

where  $\Pi$  is such that  $\Pi''(u) = P'(u)/u$ . In particular, given  $u_{\pm} > 0$ ,  $v_{\pm} \in \mathbb{R}$  and  $c \in \mathbb{R}$ , the function

$$(20) \quad (u, v)(x, t) := (u_-, v_-) \chi_{(-\infty, ct)}(x) + (u_+, v_+) \chi_{(ct, +\infty)}(x)$$

(where  $\chi_I$  is the characteristic function of the set  $I$ ) is an admissible solution to (19) if it is a weak solution and it satisfies the inequality

$$(21) \quad \partial_t \mathcal{E} + \partial_x \mathcal{Q} \leq 0$$

in the sense of distributions. Thanks to the galileian invariance, we may assume without loss of generality, the speed  $c$  to be zero. Thus, the requirement of being a weak solution translates in the classical Rankine-Hugoniot conditions, that read as

$$(22) \quad [v]_0 = 0, \quad \left[ \frac{v^2}{u} + P(u) \right]_0 = 0$$

where  $[g]_\xi := g_+ - g_-$  denotes the jump of the function  $g$  at  $x = \xi$ . Similarly, the entropy condition (21) reads as

$$(23) \quad [\mathcal{Q}]_0 = \left[ \frac{v^3}{2u^2} + \Pi'(u)v \right]_0 \leq 0.$$

Conditions (22)–(23) select the possible couples  $(w_-, w_+)$  such that jump solution (20) defines an admissible solution to (19). The admissible couples can be explicitly characterized.

**Lemma 3.1.** *Given  $P \in C^2([0, +\infty))$  such that  $P'(u), P''(u) > 0$  for any  $u \geq 0$ , let us set*

$$(24) \quad F(u, v) := \frac{v^2}{u} + P(u) \quad u > 0, v \in \mathbb{R}.$$

*Then, for any  $v_1 > 0$ , there exists  $f(v_1) > 0$  such that for any  $v_2 > f(v_1)$  equation  $F(u, v_1) = v_2$  possesses exactly two solutions  $u_\pm := u_\pm(v_1, v_2)$ . Moreover, the function  $f$  is strictly increasing and convex and such that  $f(0) = P(0)$ ,  $f'(0) = 2\sqrt{P'(0)}$ .*

*Proof.* There hold

$$\frac{\partial F}{\partial u}(u, v) = -\frac{v^2}{u^2} + P'(u), \quad \frac{\partial^2 F}{\partial u^2}(u, v) = \frac{2v^2}{u^3} + P''(u)$$

Hence, for any  $v > 0$ , the function  $F(\cdot, v)$  has a single absolute minimum point  $u_* = u_*(v)$  uniquely defined by the implicit relation

$$P'(u_*)u_*^2 = v^2.$$

Then, the function  $f = f(v)$  is defined by

$$f(v) := \min_{u>0} F(u, v) = \frac{v^2}{u_*} + P(u_*) = P'(u_*)u_* + P(u_*).$$

Moreover, differentiating we deduce

$$\frac{df}{dv} = \{P''(u_*)u_* + 2P'(u_*)\} \frac{du_*}{dv} = \frac{2v}{u_*} > 0$$

and

$$\frac{d^2 f}{dv^2} = \frac{2P''}{P''u_* + 2P'} > 0,$$

showing the stated properties of the function  $\phi$ . □

**Proposition 3.2.** *Let  $P \in C^2([0, +\infty))$  be such that  $P'(u), P''(u) > 0$  for any  $u \geq 0$ . For any couple  $(v_1, v_2)$  with  $v_1 > 0$  and  $v_2 > f(v_1)$  there exists unique  $(u_\pm, v_\pm)$  such that*

$$v_\pm = v_1, \quad F(u_\pm, v_1) = \frac{v_1^2}{u_\pm} + P(u_\pm) = v_2, \quad u_- < u_+$$

*such that the function (20) is a weak solution to (19) satisfying condition (23).*

*Proof.* Couples  $(u_\pm, v_\pm)$  connected by a single entropic stationary jump are such that  $v_-$  and  $v_+$  are equal with common value denoted by  $v_*$  and

$$F(u_-, v_*) = F(u_+, v_*).$$

The properties of the function  $F$  shows that, given  $v_*$  there is a single couple of values for which the above relation is satisfied. It only remains to analyze condition (23).

For  $v_1 > 0$ , the entropy condition becomes

$$(25) \quad \Lambda(u_+, v_1) := \frac{v_1^2}{u_+^2} + 2\Pi'(u_+) \leq \frac{v_1^2}{u_-^2} + 2\Pi'(u_-) = \Lambda(u_-, v_1).$$

Since there holds

$$\frac{\partial \Lambda}{\partial u}(u, v_1) = -\frac{2v_1^2}{u^3} + 2\Pi''(u) = \frac{2}{u} \frac{\partial F}{\partial u}(u, v_1),$$

the function  $\Lambda$  has the same monotonicity of  $F$  with growth rate that decreases when  $u$  increases with respect to the one of  $F$ . Thus, condition (25) is satisfied if and only if  $u_+ \geq u_-$ .  $\square$

As an example, in the power-law case  $P(u) = \kappa u^{\alpha+1}/(\alpha+1)$  with  $\alpha > 0$ , there holds

$$u_* = \kappa^{-\frac{1}{\alpha+2}} v^{\frac{2}{\alpha+2}}, \quad f(v) = \frac{\alpha+2}{\alpha+1} \kappa^{\frac{1}{\alpha+2}} v^{\frac{2(\alpha+1)}{\alpha+2}}$$

Note that the function  $f$  is not differentiable two times at  $u = 0$ .

**Approximate invariant manifold.** Next, we build a one-parameter family of functions  $W = W(\cdot; \xi)$  for  $\xi \in (-\ell, \ell)$  forming an approximate invariant manifold for (17) and converging as  $\varepsilon \rightarrow 0$  to

$$(26) \quad W_{\text{hyp}}(x, \xi) = (u_-, v_*)\chi_{(-\ell, \xi)}(x) + (u_+, v_*)\chi_{(\xi, \ell)}(x)$$

Recalling the general procedure presented in Section 2, we want to choose  $W$  so that the residual  $R(\xi)$  is a delta distribution concentrated at  $\xi$ . Thus, given  $\xi \in I$ , we opt for defining  $W$  by matching at  $\xi \in I$  at the state  $(u_*, v_*)$  the two stationary solutions of (17) in  $(-\ell, \xi)$  and  $(\xi, \ell)$ . Precisely, we denote by  $W_- = (U_-, V_-)$  and  $W_+ = (U_+, V_+)$  the solutions to

$$(27) \quad \frac{dv}{dx} = 0, \quad \frac{d}{dx} \left\{ \frac{v^2}{u} + P(u) - \varepsilon v(u) \frac{d}{dx} \left( \frac{v}{u} \right) \right\} = 0,$$

in  $(-\ell, \xi)$  and in  $(\xi, \ell)$  respectively, satisfying the boundary conditions

$$\begin{aligned} W_-(-\ell; \xi) &= (u_-, v_*), & W_-(\xi; \xi) &= (u_*, v_*), \\ W_+(\xi; \xi) &= (u_*, v_*), & W_+(\ell; \xi) &= (u_+, v_*), \end{aligned}$$

where  $u_*$  is the absolute minimum point of the function  $F(\cdot, v_*)$ , i.e. is uniquely determined by the requirement  $P'(u_*)u_*^2 = v_*^2$  (see Lemma 3.1).

Solutions to (27) solve

$$(28) \quad v = v_1, \quad \varepsilon \frac{\partial G}{\partial u}(u, v_1) \frac{du}{dx} = \kappa - F(u, v_1)$$

where  $v_1, \kappa$  are integration constants, the function  $F$  is defined in (24) and the function  $G$  is given by

$$G(u, v) := v \int_{\bar{u}}^u \frac{v(s)}{s^2} ds,$$

for some fixed  $\bar{u} > 0$ . From now on, let us consider  $v_1 > 0$ .

For  $F$  defined in (24) and given  $v_*$ , let us set

$$\Sigma := \{(u, \kappa) : u \in (u_-, u_+), \quad \kappa > F(u, v_*)\}$$

and define the function

$$\Gamma(u, \kappa) := \int_{u_*}^u \frac{\partial_u G(s, v_*)}{\kappa - F(s, v_*)} ds \quad \text{for } (u, \kappa) \in \Sigma.$$

Then, solutions to (27) satisfying the condition  $u(\xi) = u_*$  are implicitly defined by

$$v(x) = v_*, \quad \varepsilon \Gamma(u(x), \kappa) = x - \xi$$

Denoting by  $u_{\pm}(\kappa)$  the two solution of  $F(u, v_*) = \kappa$  (see Lemma 3.1), the function  $\Gamma$  is such that

$$\Gamma(u, +\infty) = 0, \quad \Gamma(u_-(\kappa), \kappa) = -\infty, \quad \Gamma(u_+(\kappa), \kappa) = +\infty$$

$$\Gamma(u, \cdot) \text{ is increasing if } u < u_*, \quad \Gamma(u, \cdot) \text{ is decreasing if } u > u_*.$$

As a consequence, for any  $\xi \in (-\ell, \ell)$  there exist (unique)  $\kappa_{\pm} = \kappa_{\pm}(\xi) \in (F(u_{\pm}, v_*), +\infty)$  such that

$$\varepsilon \Gamma(u_-, \kappa_-) = -\ell - \xi \quad \text{and} \quad \varepsilon \Gamma(u_+, \kappa_+) = \ell - \xi.$$

Correspondingly, we set

$$W(x, \xi) = \begin{cases} (U_-(x, \xi), v_*) & -\ell < x < \xi < \ell \\ (U_+(x, \xi), v_*) & -\ell < \xi < x < \ell, \end{cases}$$

where functions  $U_{\pm}$  are implicitly given by

$$(29) \quad \varepsilon \Gamma(U_{\pm}(x, \xi), \kappa_{\pm}) = x - \xi.$$

Next, let us calculate the residual of  $W$ : for any test function  $\Phi = (\varphi_1, \varphi_2)$ , there holds

$$\begin{aligned} \langle R(\cdot; \xi), \Phi \rangle &= \int_I \left\{ F(U, v_*) - \varepsilon \frac{\partial G}{\partial u}(U, v_*) \frac{\partial U}{\partial x} \right\} \frac{d\varphi_2}{dx} dx \\ &= \left\{ \left( F(U_-, v_*) - \varepsilon \frac{\partial G}{\partial u}(U_-, v_*) \frac{\partial U_-}{\partial x} \right) - \left( F(U_+, v_*) - \varepsilon \frac{\partial G}{\partial u}(U_+, v_*) \frac{\partial U_+}{\partial x} \right) \right\} \Big|_{x=\xi} \varphi_2(\xi) \\ &= \varepsilon \left[ \frac{\partial G}{\partial u}(U, v_*) \frac{\partial U}{\partial x} \right]_{\xi} \varphi_2(\xi) \end{aligned}$$

Differentiating (29) with respect to  $x$  we infer

$$\varepsilon \frac{\partial G}{\partial u}(U_{\pm}(x, \xi), v_*) \frac{\partial U_{\pm}}{\partial x}(x, \xi) = \kappa_{\pm} - F(U_{\pm}(x, \xi), v_*)$$

so that, with  $r = r(\xi)$  as in (10), we obtain

$$r(\xi) = (0, \kappa_+(\xi) - \kappa_-(\xi))$$

The functions  $\xi \mapsto \kappa_{\pm}(\xi)$  are the inverse of the relations

$$\xi = -\ell - \varepsilon \Gamma(u_-, \kappa_-) \quad \text{and} \quad \xi = \ell - \varepsilon \Gamma(u_+, \kappa_+),$$

thus, as a consequence of the properties of function  $\Gamma$ , the difference function  $\xi \mapsto \kappa_+(\xi) - \kappa_-(\xi)$  is monotone increasing and such that

$$\lim_{\xi \rightarrow -\ell^+} \kappa_+(\xi) - \kappa_-(\xi) = -\infty \quad \text{and} \quad \lim_{\xi \rightarrow +\ell^-} \kappa_+(\xi) - \kappa_-(\xi) = +\infty.$$

Therefore, there exists unique  $\xi_* \in (-\ell, \ell)$  such that  $(\kappa_+ - \kappa_-)(\xi_*) = 0$  and such a value is such that  $(U^{\varepsilon}(\cdot; \xi_*), v_*)$  is the unique steady state of the problem.

Finally, denoting by  $\omega_1$  the first eigenfunction of the adjoint operator  $\mathcal{L}_\xi^*$  and setting  $\omega_1(x, \xi) = (\phi(x, \xi), \psi(x, \xi))$ , equation (11) becomes

$$\frac{d\xi}{dt} = -\frac{(\phi(\xi, \xi), \psi(\xi, \xi)) \cdot (0, \kappa_+(\xi) - \kappa_-(\xi))}{(\phi(\xi, \xi), \psi(\xi, \xi)) \cdot (u_+ - u_-, 0)}$$

that gives

$$(30) \quad \frac{d\xi}{dt} = -\frac{\psi(\xi, \xi) \kappa_+(\xi) - \kappa_-(\xi)}{\phi(\xi, \xi) u_+ - u_-}.$$

The next step is to determine an appropriate (approximate) representation of the ratio  $\psi(\xi, \xi)/\phi(\xi, \xi)$ . This consists in analyzing in details the eigenvalue problem for the adjoint operator of the linearization at  $(U(x, \xi), v_*)$ .

**First eigenfunction of the adjoint operator.** As a first step, let us derive the specific expression of the linearized problem. Given  $W(x, \xi) := (U(x, \xi), v_*)$ , let us look for solution to (17) in the form  $W + (u, v)$ . The perturbation  $(u, v)$  satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial x} - \frac{\partial U}{\partial \xi} \frac{d\xi}{dt}, \\ \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x} \left\{ \rho(x) + a_1(x)u + a_2(x)v + b_1(x)\frac{\partial u}{\partial x} + b_2(x)\frac{\partial v}{\partial x} + h.o.t. \right\} \end{aligned}$$

where

$$\begin{aligned} \rho(x) &:= -F(U(x), v_*) - \varepsilon (G(U(x), v_*))' \\ a_1(x) &:= -\partial_u F(U(x), v_*) - \varepsilon \partial_{uu} G(U(x), v_*) U'(x) \\ a_2(x) &:= -\partial_v F(U(x), v_*) - \varepsilon \partial_{uv} G(U(x), v_*) U'(x) \\ b_1(x) &:= -\varepsilon \partial_u G(U(x), v_*) \\ b_2(x) &:= \varepsilon v(U(x)) U^{-1}(x) \end{aligned}$$

and *h.o.t.* collects higher order terms. Then, the linear operator  $\mathcal{L}_\xi$  is

$$\mathcal{L}_\xi \begin{pmatrix} u \\ v \end{pmatrix} = \frac{d}{dx} \left\{ \begin{pmatrix} 0 & -1 \\ a_1(x) & a_2(x) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b_1(x) & b_2(x) \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \right\}$$

Setting  $w = (u, v)$  and taking the scalar product against the line vector  $\omega = (\phi, \psi)$ , we get

$$\begin{aligned} \omega \cdot \mathcal{L}_\xi w &= \frac{d}{dx} \left\{ (\phi \ \psi) \begin{pmatrix} 0 & -1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + (\phi \ \psi) \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \right\} \\ &\quad - (\phi' \ \psi') \begin{pmatrix} 0 & -1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - (\phi' \ \psi') \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \\ &= \frac{d}{dx} \mathcal{J}[w, \omega] + (\mathcal{L}_\xi^* \omega) \cdot w \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}[w, \omega] &= (\phi \ \psi) \begin{pmatrix} 0 & -1 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &\quad + (\phi \ \psi) \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} - (\phi' \ \psi') \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

and

$$\mathcal{L}_\xi^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & -a_1(x) \\ 1 & -a_2(x) \end{pmatrix} \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} + \frac{d}{dx} \left\{ \begin{pmatrix} 0 & b_1(x) \\ 0 & b_2(x) \end{pmatrix} \begin{pmatrix} \phi' \\ \psi' \end{pmatrix} \right\}$$

The linearization of the boundary conditions (18) for the operator  $\mathcal{L}_\xi$  gives

$$(31) \quad (Uv - v_*u)(\pm\ell) = 0, \quad v(-\ell) = 0.$$

Thus, there holds

$$\begin{aligned} \mathcal{J}[w, \omega] \Big|_{x=-\ell} &= \psi(-\ell) \{b_1 u' + b_2 v'\} \Big|_{x=-\ell} = 0, \\ \mathcal{J}[w, \omega] \Big|_{x=+\ell} &= -\phi(+\ell)v(+\ell) + \psi(+\ell) \{a_1 u + a_2 v + b_1 u' + b_2 v'\} \Big|_{x=+\ell} = 0 \end{aligned}$$

and the requirement  $\mathcal{J}|_{\partial I} = 0$  is satisfied if

$$(32) \quad \phi(+\ell) = 0, \quad \psi(\pm\ell) = 0,$$

that are the boundary conditions for  $\mathcal{L}_\xi^*$ .

Next, we consider the eigenvalue problem for the adjoint operator  $\mathcal{L}_\xi^*$  with the aim of determining an approximate expression for the ratio  $\psi(\xi)/\phi(\xi)$ , where  $(\phi, \psi)$  is the eigenvector relative to the first eigenvalue  $\lambda$ . The eigenvalue problem reads as

$$\begin{cases} -a_1(x)\psi' + (b_1(x)\psi')' = \lambda\phi, \\ \phi' - a_2(x)\psi' + (b_2(x)\psi')' = \lambda\psi. \end{cases}$$

with boundary condition (32). Setting  $\theta := \psi'$  and  $w = (\phi, \psi, \theta)$ , after some standard algebraic manipulation, the above system can be rewritten as

$$(33) \quad \frac{dw}{dx} = \mathbb{A} w \quad \text{where} \quad \mathbb{A} := \begin{pmatrix} \lambda U/v_* & \lambda & a_{13} \\ 0 & 0 & 1 \\ -\varepsilon^{-1}\lambda/\partial_u G & 0 & \varepsilon^{-1}\partial_u F/\partial_u G \end{pmatrix}$$

with  $\partial_u F = \partial_u F(U, v_*)$ ,  $\partial_u G = \partial_u G(U, v_*)$  and

$$a_{13} = -\frac{U}{v_*} \left( \frac{v_*^2}{U^2} + P'(U) \right) - \varepsilon \frac{v'(U)}{U} \frac{dU}{dx}$$

In the limiting regime  $\varepsilon \rightarrow 0^+$ , the profile  $U$  tends to a step profile joining at  $x = \xi$  the values  $u_\pm$ . Thus, we consider the matrix-valued  $\mathbb{A}$  as piecewise constant and a representation for the eigenfunction can be obtained by an appropriate matching at  $x = \xi$  of solutions to (33) with  $U$  considered as a constant.

In order to determine the spectral decomposition of  $\mathbb{A}$  and its behavior in the regime  $\varepsilon \rightarrow 0$ , we analyze the characteristic roots  $\mu$ , solutions to

$$\varepsilon \det(\mathbb{A} - \mu I) = -\varepsilon \mu^3 + \left( \frac{\partial_u F}{\partial_u G} + \varepsilon \frac{\lambda U}{v_*} \right) \mu^2 + \frac{2\lambda U}{v(U)} \mu - \frac{\lambda^2}{\partial_u G} = 0$$

Plugging expansion for  $\mu$  in the form  $\mu = \mu_1 \varepsilon^{-1} + \mu_0 + o(1)$  as  $\varepsilon \rightarrow 0$  and skipping the calculations for shortness, we determine asymptotic formulas for the three roots  $\mu_0, \mu_\pm$

$$(34) \quad \mu = \mu_0 = \frac{1}{\varepsilon} \frac{\partial_u F}{\partial_u G} + o(\varepsilon^{-1}), \quad \mu = \mu_\pm = C_\pm \lambda + o(1).$$

with  $C_{\pm}$  are such that

$$\partial_u F(U, v_*) C^2 + 2 v_* U C - 1 = 0,$$

that gives, thanks to the specific form of the function  $F$ ,

$$(35) \quad C_{\pm} := \frac{2}{v_* U^{-1} \pm \sqrt{P'(U)}}.$$

Corresponding expression for the right/left eigenvectors  $r/\ell$  of  $\mathbb{A}$  relative to the eigenvalues  $\mu_{0,\pm}$  can be determined starting from the relations

$$\begin{cases} \mu r_2 - r_3 = 0, \\ \lambda r_1 + (\varepsilon \partial_u G \mu - \partial_u F) r_3 = 0. \end{cases} \quad \begin{cases} \left( \frac{\lambda U}{v_*} - \mu \right) r_1 - \frac{1}{\varepsilon} \frac{\lambda}{\partial_u G} r_3 = 0. \\ \lambda r_1 - \mu r_2 = 0, \end{cases}$$

where  $r = (r_1, r_2, r_3)$  and  $\ell = (\ell_1, \ell_2, \ell_3)$ . Taking advantage of the expansions (34), we infer for  $\mu_0$  the expression

$$r_0 = (0, 0, 1) + o(1), \quad \ell_0 = (-\lambda/\partial_u F, 0, 1) + o(1)$$

and for  $\mu_{\pm}$ , using the notation given in (35),

$$r_{\pm} = \frac{1}{1 + C_{\pm}^2 \partial_u F} (C_{\pm} \partial_u F, 1, C_{\pm} \lambda) + o(1), \quad \ell_{\pm} = (C_{\pm}, 1, 0) + o(1)$$

Thus, disregarding the  $o(1)$ -terms in the expression for the right and left eigenvectors, we may introduce the projection matrices

$$\begin{aligned} \mathbb{P}_0 &:= r_0 \otimes \ell_0 = \frac{1}{\partial_u F} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda & 0 & \partial_u F \end{pmatrix} \\ \mathbb{P}_{\pm} &:= r_{\pm} \otimes \ell_{\pm} = \frac{1}{1 + C_{\pm}^2 \partial_u F} \begin{pmatrix} C_{\pm}^2 \partial_u F & C_{\pm} \partial_u F & 0 \\ C_{\pm} & 1 & 0 \\ C_{\pm}^2 \lambda & C_{\pm} \lambda & 0 \end{pmatrix} \end{aligned}$$

Finally, we get an approximated expression for the exponential matrix of  $\mathbb{A}$

$$e^{\mathbb{A}x} \approx P_0 e^{\mu_0 x} + P_- e^{\mu_- x} + P_+ e^{\mu_+ x}.$$

Considering the solution in the sub-interval  $(-\ell, \xi)$  and taking into account the boundary condition  $\psi(-\ell) = 0$ , we deduce

$$w(\xi) = (\phi(\xi), \psi(\xi), \theta(\xi)) = e^{\mathbb{A}(\xi+\ell)}(\phi(0), 0, \theta(0)).$$

Hence, we deduce

$$\begin{aligned} \frac{\phi(\xi)}{\phi(0)} &\approx \partial_u F \left\{ \frac{C_-^2}{1 + C_-^2 \partial_u F} e^{\mu_- (\xi+\ell)} + \frac{C_+^2}{1 + C_+^2 \partial_u F} e^{\mu_+ (\xi+\ell)} \right\} \\ \frac{\psi(\xi)}{\phi(0)} &\approx \frac{C_-}{1 + C_-^2 \partial_u F} e^{\mu_- (\xi+\ell)} + \frac{C_+}{1 + C_+^2 \partial_u F} e^{\mu_+ (\xi+\ell)} \end{aligned}$$

where  $\partial_u F$  is calculated at  $(u_-, v_*)$ . For  $\lambda \rightarrow 0$ , there holds

$$\frac{\psi(\xi)}{\phi(0)} \approx \frac{C_-}{1 + C_-^2 \partial_u F} + \frac{C_+}{1 + C_+^2 \partial_u F} + o(1) = \frac{(C_- + C_+)(1 + C_- C_+ \partial_u F)}{(1 + C_+^2 \partial_u F)(1 + C_-^2 \partial_u F)} + o(1) = o(1)$$

having used the relation  $C_- C_+ = -1/\partial_u F$ . Thus, we infer from the expansion for  $\mu_\pm$ , the relation

$$\frac{\psi(\xi)}{\phi(0)} \approx \left\{ \frac{C_-^2}{1 + C_-^2 \partial_u F} + \frac{C_+^2}{1 + C_+^2 \partial_u F} \right\} (\xi + \ell) \lambda + o(\lambda) = \frac{\phi(\xi)}{\phi(0)} \frac{(\xi + \ell)}{\partial_u F} \lambda + o(\lambda)$$

Hence, we end up with the asymptotic expression

$$\frac{\psi(\xi)}{\phi(\xi)} \approx \frac{\xi + \ell}{\partial_u F(u_-, v_*)} \lambda.$$

Plugging into (30), we deduce the new form

$$(36) \quad \frac{d\xi}{dt} \approx -\frac{(\xi + \ell) \lambda_1(\xi)}{\partial_u F(u_-, v_*)} \frac{\kappa_+(\xi) - \kappa_-(\xi)}{u_+ - u_-}.$$

where  $\lambda_1(\xi)$  denotes the first eigenvalue of the operator  $\mathcal{L}_\xi$ .

Note that, since  $\partial_u F(u_-, v_*) < 0$  and  $\kappa_+ - \kappa_-$  is an increasing function, the equilibrium point  $\xi_*$  is stable if and only if  $\lambda_1(\xi) < 0$ , as it should be.

**Further simplification of the slow motion equation.** As a last step, we derive an approximate expression for the functions  $\kappa_\pm$  showing that the motion is indeed exponentially slow. Such functions are implicitly defined by the relations

$$\int_{u_*}^{u_\pm} \frac{\partial_u G(s, v_*)}{\kappa_\pm - F(s, v_*)} ds = -\frac{\xi \mp \ell}{\varepsilon}$$

In the limit  $\varepsilon \rightarrow 0^+$ , the righthand side blows up, hence  $\kappa_\pm$  is such that

$$\lim_{\varepsilon \rightarrow 0^+} \kappa_\pm(\xi) = F(u_\pm, v_*).$$

To get a preciser description of  $\kappa_\pm$  for small  $\varepsilon$ , we approximate functions  $F$  and  $\partial_u G$  by

$$F(s, v_*) \approx F(u_\pm, v_*) + \partial_u F(u_\pm, v_*)(s - u_\pm),$$

$$\partial_u G(s, v_*) \approx \partial_u G(u_\pm, v_*)$$

Setting  $h_\pm(\xi) := \kappa_\pm(\xi) - F(u_\pm, v_*)$ , we obtain

$$\int_{u_*}^{u_\pm} \frac{\partial_u G(u_\pm, v_*)}{h_\pm(\xi) - \partial_u F(u_\pm, v_*)(s - u_\pm)} ds = -\frac{\xi \mp \ell}{\varepsilon}$$

Integrating, it follows

$$\frac{\partial_u G(u_\pm, v_*)}{\partial_u F(u_\pm, v_*)} \ln \left( 1 - \frac{\partial_u F(u_\pm, v_*)(u_* - u_\pm)}{h_\pm(\xi)} \right) = -\frac{\xi \mp \ell}{\varepsilon}$$

from which we infer

$$h_-(\xi) = -\frac{\partial_u F^-(u_* - u_-)}{\exp \left\{ -\frac{\partial_u F^-}{\partial_u G^-} \frac{\xi + \ell}{\varepsilon} \right\} - 1}, \quad h_+(\xi) = \frac{\partial_u F^+(u_+ - u_*)}{\exp \left\{ \frac{\partial_u F^+}{\partial_u G^+} \frac{\ell - \xi}{\varepsilon} \right\} - 1}$$

where  $\partial_u F^\pm = \partial_u F(u_\pm, v_*)$  and  $\partial_u G^\pm = \partial_u G(u_\pm, v_*)$ . Finally, since  $F(u_-, v_*) = F(u_+, v_*)$ , we obtain

$$\kappa_+(\xi) - \kappa_-(\xi) \approx \partial_u F^+(u_+ - u_*) \exp \left\{ -\frac{\partial_u F^+}{\partial_u G^+} \frac{\ell - \xi}{\varepsilon} \right\} + \partial_u F^-(u_* - u_-) \exp \left\{ \frac{\partial_u F^-}{\partial_u G^-} s \frac{\xi + \ell}{\varepsilon} \right\}$$



as  $\varepsilon \rightarrow 0$ . Inserting into (36), we obtain the equation

$$(37) \quad \frac{d\xi}{dt} = - \left\{ \frac{\partial_u F^+}{\partial_u F^-} \frac{u_+ - u_*}{u_+ - u_-} \exp \left( - \frac{\partial_u F^+}{\partial_u G^+} \frac{\ell - \xi}{\varepsilon} \right) + \frac{u_* - u_-}{u_+ - u_-} \exp \left( \frac{\partial_u F^-}{\partial_u G^-} \frac{\xi + \ell}{\varepsilon} \right) \right\} (\xi + \ell) \lambda_1(\xi)$$

which is the ultimate version of the equation for the dynamics along the approximate manifolds.

#### 4. CONCLUSIONS

Equation (37) describes the motion's equation along the approximate manifold  $\{W(\cdot, \xi)\}$  and it should be considered as an equation for the movement of the transition layer from  $(u_-, v_*)$  to  $(u_+, v_*)$  in the bounded interval  $(-\ell, \ell)$  due to the boundary conditions. To our knowledge, this is the first attempt of determining a relation that should be capable to quantify the boundary effects in the case of isentropic Navier–Stokes equation for compressible fluids. Starting from it, a number of possible issues amenable to future investigations arise. We propose a list of eventual topics, dictated by our personal taste.

1. The determination of the size and, mainly, the sign of the first eigenvalue  $\lambda_1$  of the linearized operator  $\mathcal{L}_\xi$  has not yet been explored. In particular, its sign is fundamental to determine stability of the single steady state. Comparing with the case of scalar Burgers equation, it is expected that asymptotic stability holds and thus that  $\lambda_1$  is negative.

2. The manifold builded to determine the form of the motion's equation is approximated. The residual is in any case exponentially small and thus the dynamics remain confined in a small neighborhood around such manifold. In analogy with the results proved for the Burgers equation, it would be interesting to analyze if there exists an exact invariant manifold, lying close to the manifold of the approximate solutions, and determine a corresponding equation for the dynamics along it.

3. The equation for the location  $\xi$  of the transition layer is almost explicit. The unique term remaining unknown is the first eigenvalue  $\lambda_1$  that could be determined numerically. We wonder if it would be possible to verify numerically the reliability of the equation, bypassing the smallness of the right hand side and, thus, the very long timescale of the shifting phenomenon.

4. As shown at the beginning of Section 2, the approach is formal but flexible and, in principle, capable to be applied to different systems fitting into the general class of hyperbolic–parabolic systems. Among others, the extension of the approach to the case of non-isentropic Navier–Stokes would be particularly intriguing.

5. Following results available in the literature (see Introduction), the case of the half-line could also be taken into account. In particular, to quantify by means of a motion's equation the effect of the presence of a single boundary seems to be a natural variant of the case considered here

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